國立彰化師範大學105學年度第1學期學士班轉學生招生考試試題

系所：數學系　　　　年級：二年級

科目：線性代數

☆☆請在答案紙上作答☆☆　　　　

1. Let $A=\begin{bmatrix} 3 & 1 & 2 & 2 \\ -1 & 0 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}$. Determine the rank and nullity of A and find a basis for each of the following subspaces of $\mathbb{R}^4$: (20%)

   (1) null space of A;

   (2) column spaces of A and A^T;

   (3) row space of A and A^T.

2. Let V be a vector space with basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. Let $\mathbf{w}_1=\mathbf{v}_1+\mathbf{v}_2$, $\mathbf{w}_2=\mathbf{v}_1-\mathbf{v}_2$, $\mathbf{w}_3=\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3$. Prove: for every $\mathbf{v}\in V$, there exist a unique set of scalars c_1,c_2,c_3 such that $\mathbf{v}=c_1\mathbf{w}_1+c_2\mathbf{w}_2+c_3\mathbf{w}_3$. (20%)

3. Let $A_n=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \end{bmatrix}$, $n\ge 3$. For example, $A_3=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$, $A_4=\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. Prove that $|A_n|=0$. (10%)

4. Let $A=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. (15%)

   (1) Find the eigenvalues of A.

   (2) Find an invertible matrix C such that $C^{-1}AC$ is a diagonal matrix.

5. Find an orthonormal basis for the subspace $W = sp([1,2,0,2],[2,1,1,1],[1,0,0,1])$ of $\mathbb{R}^4$. (10%)

6. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x_1,x_2,x_3) = (x_1+x_2+x_3, x_1+x_2, x_3)$. (15%)

   (1) Find the standard matrix representation A of T, and

   (2) Find the matrix representation R_B of T relative to the ordered bases $B = ([1,0,1],[1,1,0],[0,1,1])$.

7. Let A be an $n \times n$ matrix. Prove that, if $v_1, v_2, \ldots, v_n$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, then the set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. (10%)