國立彰化師範大學115學年度碩士班招生考試試題

系所：數學系(選考乙)、統計資訊研究所(選考乙)　　科目：統計學

☆☆請在答案紙上作答☆☆　　

(18%) Let X be a continuous random variable with pdf $f(x) = c(x - x^2),\ 0 < x < 1$.

(1) (6%) Find the constant c.

(2) (6%) Find the probability $P(0.6 < x < 1.2)$.

(3) (6%) Find the median of this distribution.

1. (20%) Let X and Y have a uniform distribution on the set of points with integer coordinates in $S = \{(x, y): 0 \le x \le 7, x \le y \le x + 2\}$. That is, $f(x, y) = 1/24$, $(x, y) \in S$, and both x and y are integers.

(1) (6%) Find the marginal pmf of X, $f_X(x)$.

(2) (6%) Find the conditional pmf of Y, given that $X = x$, $f_{Y|x}(y|x)$.

(3) (8%) Find the conditional expectation $E(Y|x)$.

3. (12%) Let X have the uniform distribution $U(-1,3)$. Find the pdf of $Y = X^2$.

4. (18%) Suppose $X_1, X_2, \dots, X_n$ is a random sample (i.e., independent and identically distributed) from a normal population $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. The sample mean and sample variance are denoted by $\overline{X}$ and S^2, respectively.

(1) (8%) Define $Y = (n-1)S^2/\sigma^2$. Derive the distribution of Y. In your response, provide the complete mathematical derivation, state the name of the resulting distribution, and specify its parameter(s).

(2) (5%) Let $W = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$. Identify the distribution of W. State the necessary condition regarding the relationship between $\overline{X}$ and S^2 for this result to be valid.

(3) (5%) A researcher collects a random sample of size n=16 from this population, yielding $\overline{X}$=50 and S^2=25. If the researcher claims that the population mean μ is greater than 47, test this claim at the α= 0.05 significance level. Your answer must include the null hypothesis (H_0), the alternative hypothesis (H_1), the calculated test statistic, and a conclusion.

(*Note*: $Z_{0.025}$=1.96, $t_{0.025,\, df=15}$=2.131, $t_{0.025,\, df=16}$=2.120, $Z_{0.05}$=1.645, $t_{0.05,\, df=15}$=1.753, $t_{0.05,\, df=16}$=1.746)

5. (12%) Consider a hierarchical model where a random sample $X_1, X_2, \dots, X_n$ is generated from a normal population such that the population mean is a random variable Θ. Specifically, $\Theta \sim N(\mu, \tau^2)$ represents the variability of the population mean, and given $\Theta = \theta$, the observations $X_1, X_2, \dots, X_n$ are conditionally i.i.d. $N(\theta, \sigma^2)$. Furthermore, let σ^2 and τ^2 be known positive constants.

(1) (6%) Identify the marginal distribution of a single observation X_i. Specify the name of the distribution and its corresponding parameters.

(2) (6%) Derive the Maximum Likelihood Estimator (MLE) of μ based on the observed sample X_1, $X_2, \ldots, X_n$.

6. (20%) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Poisson distribution with parameter $\lambda>0$.

(1) (3%) Examine whether the Poisson distribution belongs to the exponential family.

(2) (3%) Identify a complete sufficient statistic for λ based on the given random sample.

(3) (6%) Calculate the Fisher Information for the random sample $X_1, X_2, \ldots, X_n$ and determine the Cramér-Rao Lower Bound (CRLB) for the variance of any unbiased estimator of λ.

(4) (8%) Let $g(\lambda)=e^{-\lambda}$. Find the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of $g(\lambda)$ and justify your answer.