國立彰化師範大學113學年度碩士班招生考試試題

系所：數學系(選考甲)、統計資訊研究所(選考甲)　　科目：線性代數

☆☆請在答案紙上作答☆☆　　共1頁，第1頁

For all the problems, justify your steps fully.

1. (20%) Let $Ax = b$ denote the system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 5 \\ \quad\quad x_2 \quad\quad + 2x_4 = 3. \end{cases}$$

Find the solution set of $Ax = b$.

2. (20%) Find a subset β of $S = \{(1,0,-2), (2,-3,5), (8,-12,20), (0,2,-1), (7,2,0)\}$ such that β is a basis for $\text{span}(S)$.

3. (20% = 10% + 10%) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(A) Find a unitary matrix P and a diagonal matrix D such that $A = PDP^*$.

(B) Find A^{20}.

4. (20% = 10% + 10%)

(A) Given $a_0, a_1, \dots, a_n \in \mathbb{R}$, consider a Vandermonde matrix $A = \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{bmatrix}$.

Show that $\det(A) = \prod_{0 \le i < j \le n} (a_j - a_i)$.

(B) Compute $\det\left(\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \\ 1 & 6 & 36 & 216 & 1296 \end{bmatrix}\right)$.

5. (20%) Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space over $\mathbb{R}$, and let $F: V \to \mathbb{R}$ be a linear transformation. Show that there exists a unique vector $v \in V$ such that $F(u) = \langle u, v \rangle$ for all $u \in V$.