國立彰化師範大學106學年度碩士班招生考試試題

系所：數學系　　組別：甲組　　科目：機率與統計

☆☆請在答案紙上作答☆☆　　

1. Suppose $X_1, X_2, \cdots$ are independent random variables with $X_n \sim \text{Bin}(n, \frac{1}{n})$, $n = 1,2,\cdots$. Prove that $\text{P}(X_n \geq 2) \to 1 - 2e^{-1}$ as $n \to \infty$. (20%)

2. Suppose $X \sim \text{U}(-2,1)$. Let $Y = \frac{X^2}{4}$. Find the distribution of Y. (15%)

3. There are n coins in a box. When flipped, the ith coin will turn up heads with probability $\frac{i}{n}$, $i = 1,2,\cdots,n$. A coin is randomly selected from the box and is then repeatedly flipped. What is the probability that the first two flips both result in heads? (15%)

4. Let $X_1, X_2, \cdots, X_n$ be a random sample from Unif$(0,\theta)$, $\theta > 0$. Determine the MLE $\hat{\theta}$ of θ. Is $\hat{\theta}$ an uniformly minimum variance unbiased estimate of θ? Why? Can we adjust $\hat{\theta}$ to be an UMVUE of θ? (15%)

5. Let $X_1, X_2, \cdots, X_n$ be a random sample with strictly increasing population distribution function F, and let $X_{(k)}$ be the k-th order statistic of the X_i', $1 \le i \le n$. For $0 < p < 1$, let x_p be the p-th quantile of F. (20%)

(a) Show that $[X_{(i)}, X_{(j)}]$ is a confidence interval for x_p with confidence level $\sum_{k=i}^{j-1} \binom{n}{k} p^k (1-p)^{n-k}$, $1 \le i < j \le n-1$. This probability is referred to as probability of converge of x_p.

(b) Find the coverage probability for the interval $[X_{(4)}, X_{(6)}]$ for the median of the sample provided $n = 10$.

6. Consider two independent random samples $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ with variances σ_1^2 and σ_2^2, respectively, where $m, n \ge 2$. Define $S_1^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / n$ and $S_2^2 = \sum_{j=1}^{m} (Y_j - \bar{Y})^2 / m$. (15%)

(1) If we'd like to use F-testing to test the hypothesis $H_0: \sigma_1 = \sigma_2$ v.s. $H_1: \sigma_1 \ne \sigma_2$, what additional assumptions should we add to the distributions of the two independent random samples ?

(2) If n = 10, m = 8, $s_1^2 = 25, s_2^2 = 32$,
what is the outcome of the hypothesis testing in problem (a) ?